Careful Pendulum Measurement of g (to be revised next week)

In the Monday lab, we repeated a pendulum period measurement done by groups the previous week. This time all students timed a single pendulum for 100 swings. This was also done on Wednesday using the same pendulum, but with better protocol – allowing better visibility of the pendulum extremes, and checking that no transcription errors occurred. The results for both days are shown on an attached page.

Data

To accurately know the length from the pendulum pivot point to the center of the mass, we needed the following data:

distance from the bottom of the ceiling strut to pivot point = 15 mm
thickness of ceiling support over which tape measure was hooked = 1 mm
distance from ceiling support top to point on weight brass ring where wire was tied = 2670 mm
(The tape measure was hooked over the 1-mm thick ceiling strut.)
diameter of weight = 34 mm so radius of weight r_{weight} = 17 mm
height of brass ring tie point above top of weight = 5 mm
distance from center of weight to hang point on brass ring = 17 mm + 5 mm = 22 mm
Distance from pivot point to center of weight l = 2670 mm – 15 mm – 1 mm + 22 mm = 2676 mm

mass/length of support wire $m_{\text{wire}}/l_{\text{wire}}$ = $4.68\times10^{-4}\,\text{kg/m}$
length of wire is 2676 mm except that at the very top a paper clip is used with the pivot.
(The mass difference between the paper clip and wire can be ignored because the paper clip is located at the pivot point and therefore contributes negligibly to the inertia and gravitational torque.)
mass of wire $m_{wire}=(2.670\,\text{m})\cdot(4.68\times10^{-4}\,\text{kg/m})=1.25\times10^{-3}\,\text{kg}$

mass of large weight m_{weight} = 233 g

Theory of Pendulum Including Wire Inertia and Weight Rotation

$$\text{inertia of sphere about its center}\quad I_{\text{weight}}=\frac{2}{5}m_{\text{weight}}r_{\text{weight}}^2$$

$$\text{inertia of wire about its end}\quad I_{\text{wire}}=\frac{1}{3}m_{\text{wire}}l_{\text{wire}}^2$$

$$T=2\pi\sqrt{\frac{\text{inertia}}{\text{torque}}}=2\pi\sqrt{\frac{(\frac{1}{3}m_{wire}+m_{\text{weight}})l_{wire}^2+\frac{2}{5}m_{\text{weight}}r_{\text{weight}}^2}{(\frac{1}{2}m_{wire}+m_{\text{weight}})g\,l_{wire}}}$$

$$T=2\pi\sqrt{\frac{(\frac{0.00125\,\text{kg}}{3}+0.233\,\text{kg})\cdot(2.670\,\text{m})^2+\frac{2}{5}\cdot(0.233\,\text{kg})\cdot(0.017\,\text{m})^2}{(\frac{0.00125\,\text{kg}}{2}+0.233\,\text{kg})g(2.670\,\text{m})}}$$

$$T=2\pi\sqrt{\frac{(0.233417\,\text{kg})\cdot(7.1289\,\text{m}^2)+0.4\cdot(0.233\,\text{kg})\cdot(0.00029\,\text{m}^2)}{(0.233625\,\text{kg})\cdot(2.670\,\text{m})\,g}}$$

$$T=2\pi\sqrt{\frac{7.1227\,\text{m}}{g}}\quad\text{so that}\quad g=4\pi^2\frac{2.6688\,\text{m}}{T^2}=\frac{105.36\,\text{m}}{T^2}$$

The simple formula gives

$$g=\frac{4\pi^2 l}{T^2}=\frac{4\pi^2 2.670\,\mathrm{m}}{T^2}=\frac{105.41\,\mathrm{m}}{T^2}$$

So the corrections turn out to be 0.05%, mainly caused by the inertia of the wire. When we use the simple formula our values of g will be 0.05% too large or about 0.005 m/s^2 too large.

There is also a correction for the swing angle that gives

$$T=2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\theta_0^2}{16}+\frac{11}{3072}\theta_0^4+\ldots\right)$$ where θ_0 is the angle of swing in radians.

Solving this for *g* gives

$$g=\frac{4\pi^2 l}{T^2}\left(1+\frac{\theta_0^2}{16}+\ldots\right)^2=\frac{4\pi^2 l}{T^2}\left(1+\frac{\theta_0^2}{8}+\ldots\right)$$

Our pendulum had a swing distance of about 250 mm and therefore an angle of

$$\theta_0\approx\frac{250\,mm}{2670\,mm}\approx 0.094\text{ radians}$$

so that the first correction term for *g* is

$$\frac{\Delta g}{g}\approx\frac{(0.094\,rad)^2}{8}=1.1\times 10^{-3}\approx 0.11\,\% \quad .$$

This correction means that our value for *g* using only the simple formula is about 0.11% or 0.011 m/s^2 too small. It would have been wiser to use about half that swing angle and to carefully measure its swing distance.

Uncertainty Analysis

The uncertainty of the length was about $\pm 1\,\mathrm{mm}$ = $\frac{\pm 0.001\,\mathrm{m}}{2.676\,\mathrm{m}}=0.00037=0.037\,\%$

The standard deviation for the times were

Monday: $0.53\,\mathrm{s}/100=0.0053\,\mathrm{s}=\frac{\pm 0.0053\,\mathrm{s}}{3.2845\,\mathrm{s}}=0.0016=0.16\,\%$

Wednesday: $0.14\,\mathrm{s}/100=0.0014\,\mathrm{s}=\frac{\pm 0.0014\,\mathrm{s}}{3.2912\,\mathrm{s}}=0.00043=0.043\,\%$

Since the acceleration of gravity, $g=\frac{4\pi^2 l}{T^2}$, depends on the inverse square of the period, the period uncertainty contributes double its percentage to the uncertainty in the *g* value.

An easy way to see this is to consider a number like 100 with an uncertainty of 2% and check the percentage errors of 100+2 vs. 100 when squared. $(100+2)^2$ = 10404 vs. 100^2=10000. The percentage error of these squares is (10404 – 10000)/10000 = 0.0404 = 4.04%, close to double the 2% error of the 100 value. It turns out to not matter whether that the T^2 is the numerator or divisor in the equation.

The proper way to combine the length uncertainty and period squared uncertainty is as the square root of the sum of the squares of the uncertainties:

Monday's fractional error in g value = $\sqrt{0.00037^2+2*0.0016^2}=0.00229=0.23\,\%$

So Monday's g value is $9.793(1\pm 0.0023)=9.793\pm 0.023\,\mathrm{m/s^2}$

Wednesday's fractional error in g value = $\sqrt{0.00037^2+2*0.00043^2}=0.00070=0.07\,\%$

So Wednesday's g value is $9.753(1 \pm 0.00070) = 9.753 \pm 0.007\,\mathrm{m/s^2}$

The larger uncertainty of a value needs to lead to less weight (importance) for that value and it turns out the the inverse square of the fractional uncertainty should be used. For the Monday value, this is

$w_{\text{Monday}} = \frac{1}{0.0023^2} = 1.89 \times 10^5$ and for Wednesday's value, this is $w_{\text{Monday}} = \frac{1}{0.0007^2} = 20.41 \times 10^5$.

We then "normalize" the weights by dividing these by the sum of both weights getting

$$\bar{w}_{\text{Monday}} = \frac{1.89 \times 10^5}{1.89 \times 10^5 + 20.41 \times 10^5} = 0.085 \qquad \bar{w}_{\text{Monday}} = \frac{20.41 \times 10^5}{1.89 \times 10^5 + 20.41 \times 10^5} = 0.915$$

This tells us that the Wednesday data should be treated as about 10 times more valuable than the Monday value when a final average is performed. The result is

$$g_{\text{final}} = 0.085 \times 9.792\,\mathrm{m/s^2} + 9.753 \times 0.915\,\mathrm{m/s^2} = 9.756\,\mathrm{m/s^2}$$

Since the two groups used the same length, the length errors do not get reduced by averaging the results of Monday and Wednesday groups. The uncertainty of that length was simply guessed to be $\pm 1\,\mathrm{mm}$. The main difference between the Monday and Wednesday measurements was that the protocol for the Wednesday measurement had the students independently using their eyes to judge the starting and stopping points of the 100 swings rather than the instructor saying "0" and "100". Any error in the instructor's call would be a systematic error since it affects all student measurements. **Systematic errors** and are not reduced by the averaging; averaging only reduces **random errors**.