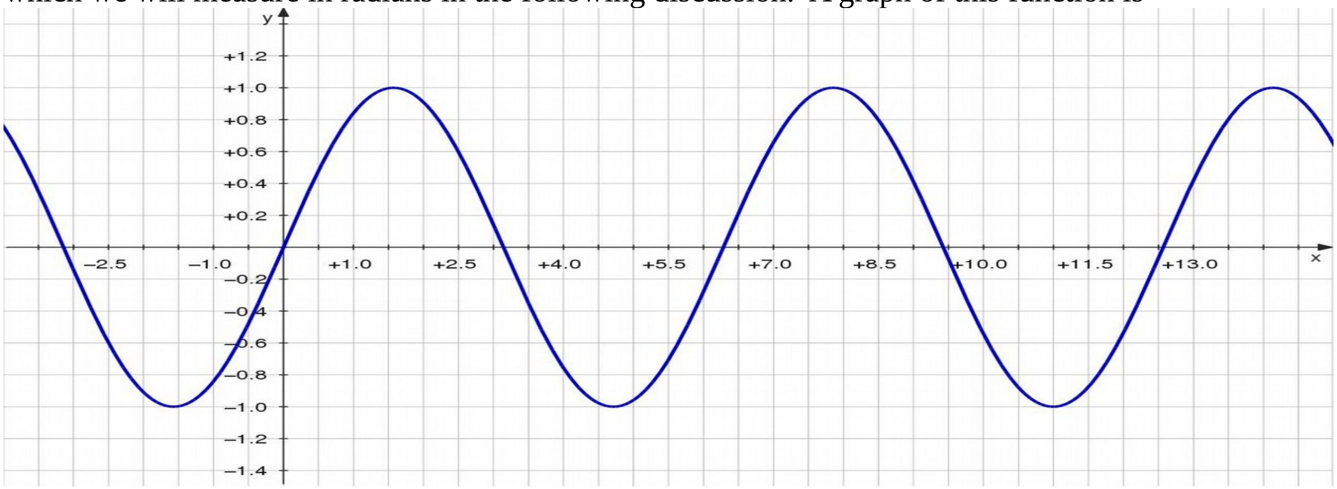
Waves and Uncertainty Principle

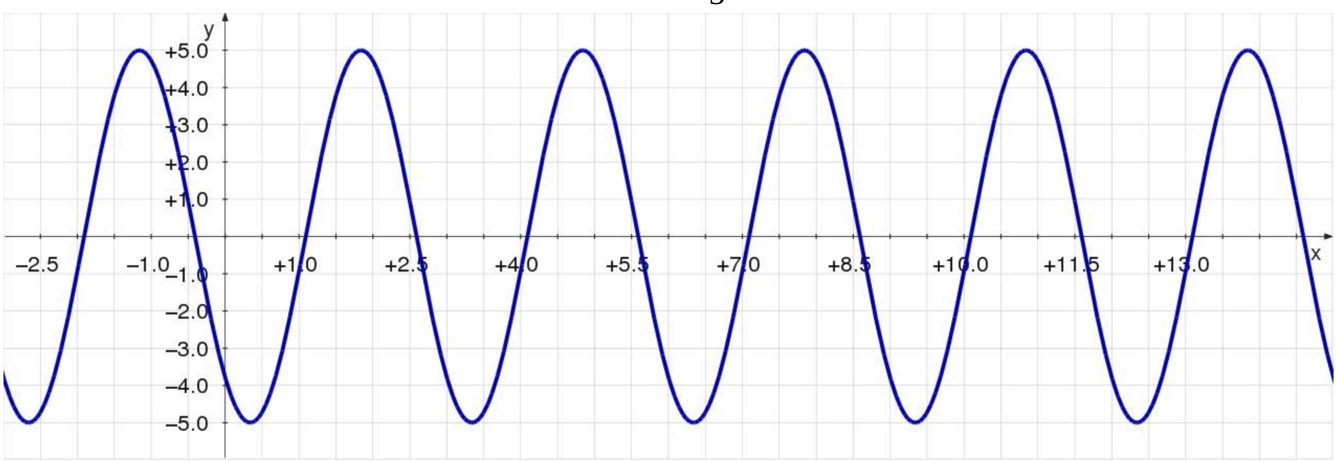
Basic Wave Equation and Related Parameters

Understanding wave phenomena is key to understanding much of physics. Water waves, radio waves, light waves, sound waves, and quantum probability waves all share concepts described in these notes.

A simple single-frequency wave can be represented by the sine curve: $y = \sin(x)$ where *x* is an angle which we will measure in radians in the following discussion. A graph of this function is



A more general formula $y = A \sin(2\pi \frac{x}{\lambda} + \phi)$ allows for specification of the **amplitude** *A*, **initial phase** φ, and **wavelength** λ. A graph of $y = 5 \sin(2\pi \frac{x}{3} + 4)$ is



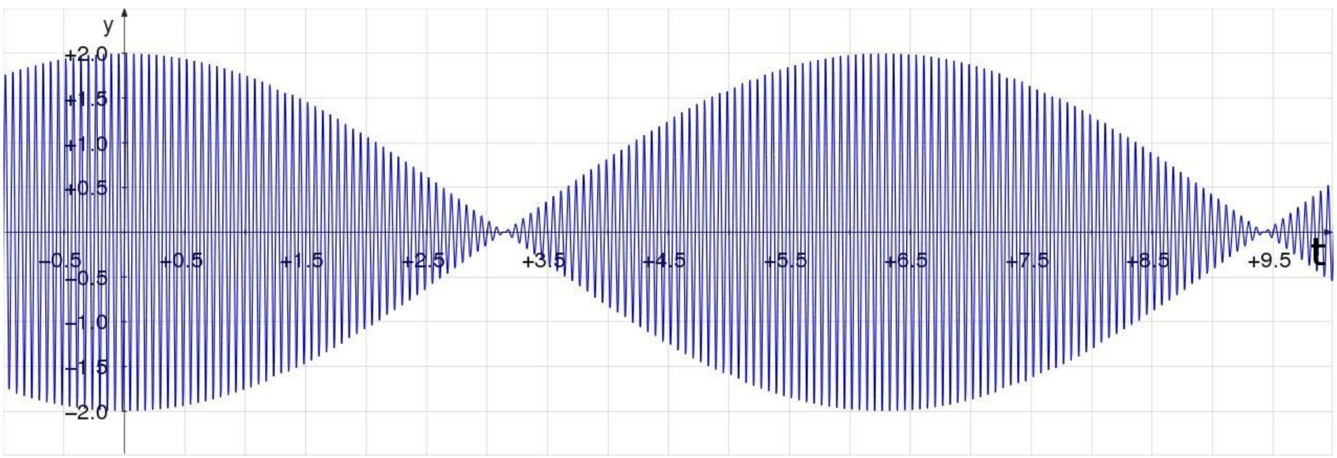
This time the sine curve peaks at ± 5 *y*-units, repeats every 3 *x*-units, and has a value $y = 5 \sin(4) = -3.78$ at $x = 0,\ 3,\ 6,\ \ldots$.

Now, we will allow for the phase to increase linearly with time *t* leading to a periodic behavior with a **period** *T* and an initial value of ϕ_0 : $\phi = 2\pi \frac{t}{T} + \phi_0$ Our sine function will then vary with both *x* and *t*: $y = A \sin(2\pi \frac{x}{\lambda} + 2\pi \frac{t}{T} + \phi_0)$. At any particular *x* position, *y* will be periodic in time with a frequency $f = 1/T$. As time moves forward, the entire sine wave as plotted vs. x will look like it is moving backwards. At any particular time, *y* will be periodic in space with the wavelength λ.

Physicists like their waves to move forward, not backward with time, so they actually introduce the time term with a negative sign. They also get tired of writing these 2π's and use two new variables, **wave-number** $k = \frac{2\pi}{\lambda}$ and **angular frequency** $\omega = \frac{2\pi}{T}$. As a result, the general wave equation is usually written as $y = A\sin(kx - \omega t + \phi_0)$. This wave will move forward at a speed of $v = \omega/k$. Its equation can then be rewritten as $y = A\sin[k(x - v\cdot t) + \phi_0]$ which more clearly illustrates the forward motion of the wave.

Beat Frequency

When two waves of the same amplitude and nearly the same frequency are combined, a beat note is apparent that has a frequency equal to the difference in their frequencies. For example, here is a graph of $y(t) = \sin(100t) + \sin(101t)$:



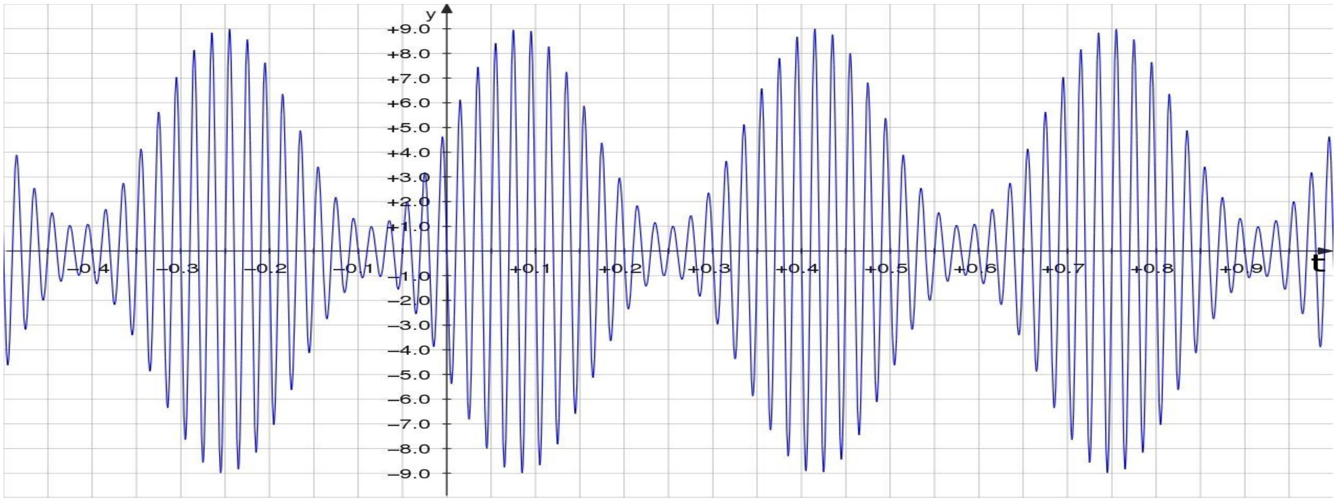
When listening to a beat frequency, you will hear the two tones periodically reinforce and cancel each other; you do not actually hear a tone at the beat frequency.

Amplitude Modulation

A single frequency wave carries no information, other than its fixed phase, frequency, and wavelength; it cannot convey music. We can, however, use a special circuit (**modulator**) to vary the amplitude, frequency, or phase of the wave in a manner that reflects the varying intensity of the music.

The following figure shows what happens when a much lower frequency tone (the **modulation frequency**) is used to vary the amplitude of a much higher frequency wave (the **carrier frequency**). The graph shows the equation $y = \left(A_0 + A_m\sin(\omega_m t)\right)\sin(kx - \omega t + \phi_0)$ vs. *t* at $x = 0$ with, $A_0 = 5$ $A_m = 4$, $\omega_m = 6\pi$, $\omega = -100\pi$, and $\phi = 0$, i.e. $y = \left(5 + 4\sin(6\pi t)\right)\sin(100\pi t)$

(The curve shown below actually is for $\varphi = \pi$, but the following discussion is for $\phi = 0$.)



A device (**demodulator**) that extracts the lower frequency tone from this wave is easily constructed. This process can therefore carry information about the tone. If instead of this constant tone, the varying tones of music or voice were used for the modulation, we would have a method to send music or voice information over the carrier wave. This is how AM (**amplitude modulation**) radio works.

Using the trigonometric identities $\sin\theta\sin\phi = \dfrac{\cos(\theta-\phi)-\cos(\theta+\phi)}{2}$, $\cos(-\theta)=\cos(\theta)$, and $\cos(\theta)=\sin(\theta+\frac{\pi}{2})$, we can rewrite the equation for this tone-modulated wave as follows:

$$\begin{aligned} y &= (5+4\sin(6\pi t))\sin(100\pi t) = 5\sin(100\pi t) + 4\sin(6\pi t)\sin(100\pi t) \\ &= 5\sin(100\pi t) + 2[\cos(6\pi t - 100\pi t) - \cos(6\pi t + 100\pi t)] \\ &= 5\sin(100\pi t) + 2\sin(94\pi t + \frac{\pi}{2}) - 2\sin(106\pi t + \frac{\pi}{2}) \end{aligned}$$

We see that this modulated wave is the sum of three pure sine waves, one at the carrier frequency of 50 Hz and, two others called **sidebands** that are 3 Hz below and above the carrier and are 2/5 as strong. A graph of this sum of three sine waves will be identical to the previous curve.

Music or voice modulation would produce constantly varying sidebands with a range of frequencies extending up to 20 kHz above and below the carrier. The upper and lower sidebands carry identical information and the carrier is simply a reference frequency for interpreting the sidebands.

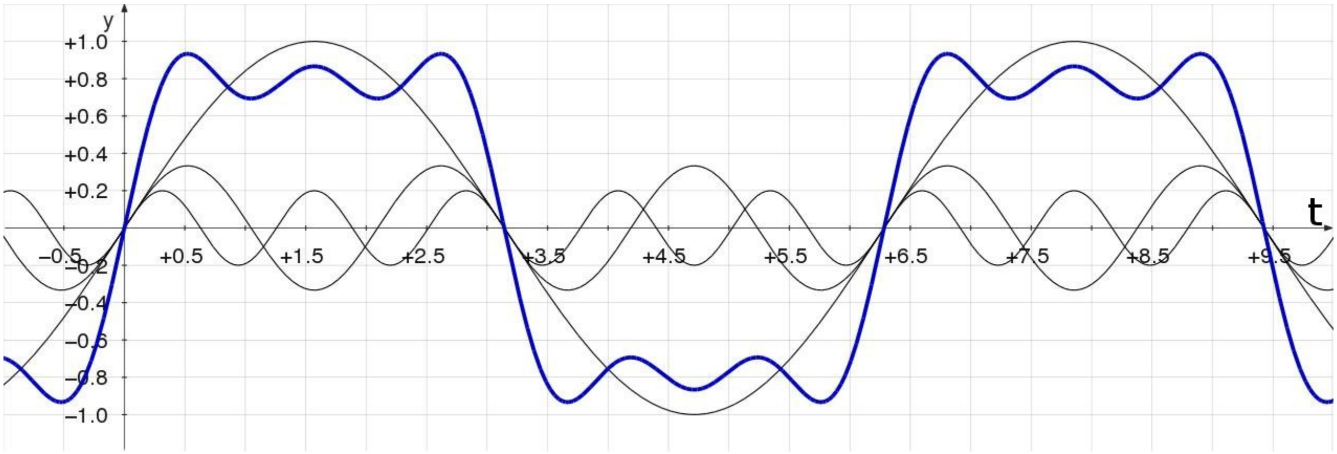
Since the modulated information is carried by the sidebands and is duplicated on either side of the carrier, some specialized communication systems put all their transmitted power into one sideband and none into the carrier or other sideband. A suitably-designed receiver can replace the carrier and extract this "single-sideband" modulation.

This example illustrates and important concept about waves: For a fixed value of *x*, waves can be fully characterized by specifying their strength vs. time *y(t)* (**time domain**) or by the strength and phase of their frequency components *Y(f)* and *Φ(f)* (**frequency domain**). In the above example *Y*=0 and *Φ*=0 at all frequencies except that *Y*(50 Hz) = 5, *Φ*(50 Hz) = 0, *Y*(47 Hz) = 2, *Φ*(47 Hz) = π /2, *Y*(53 Hz) = –2, and *Φ*(53 Hz) = π /2.

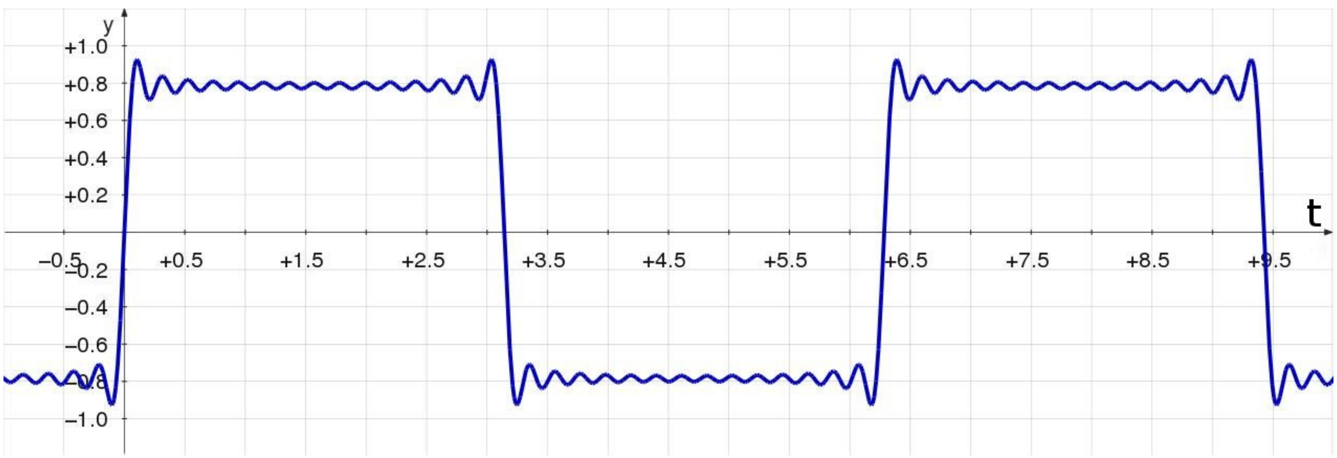
Frequency modulation and phase modulation have similar characteristics, but lead to more complicated spectra.

Fourier Series

Let's now focus again on periodic waves at $x=0$ Any function that is periodic with period $T=\frac{2\pi}{\omega}$ can be closely approximated by a sum of sine waves with periods that are integer multiples of T and and have carefully chosen amplitudes and phases. How this comes about is illustrated by the following graph that shows the function $f(t)=\sin(t)+1/3\sin(3t)+1/5\sin(5t)$ which approximates a square wave that has a period of 2π.



If more terms are added following the same pattern, the approximation gets better. Here is the result using 15 terms extending out to $\frac{1}{29}\sin(29t)$:



With an infinite number of terms, this would become a perfect square wave except for an infinitesmally narrow 9% overshoot at each jump. (Functions without discontinuous jumps are represented without such overshoots.)

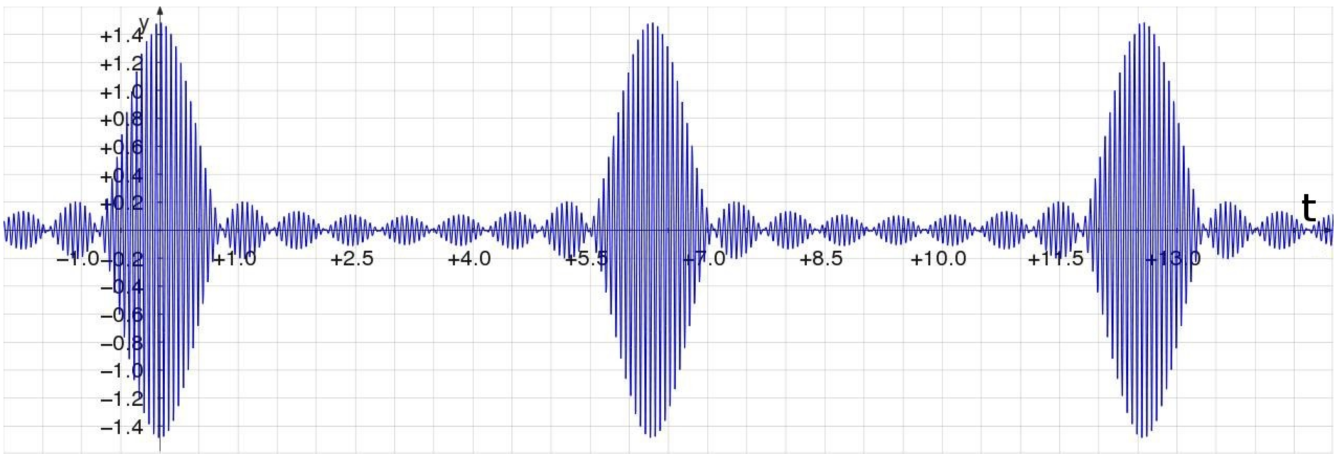
Sawtooth waves, triangular waves, and in fact, periodic waves of any shape may be constructed in a similar manner. These series of sine functions are called Fourier sine series.

Wave Packets

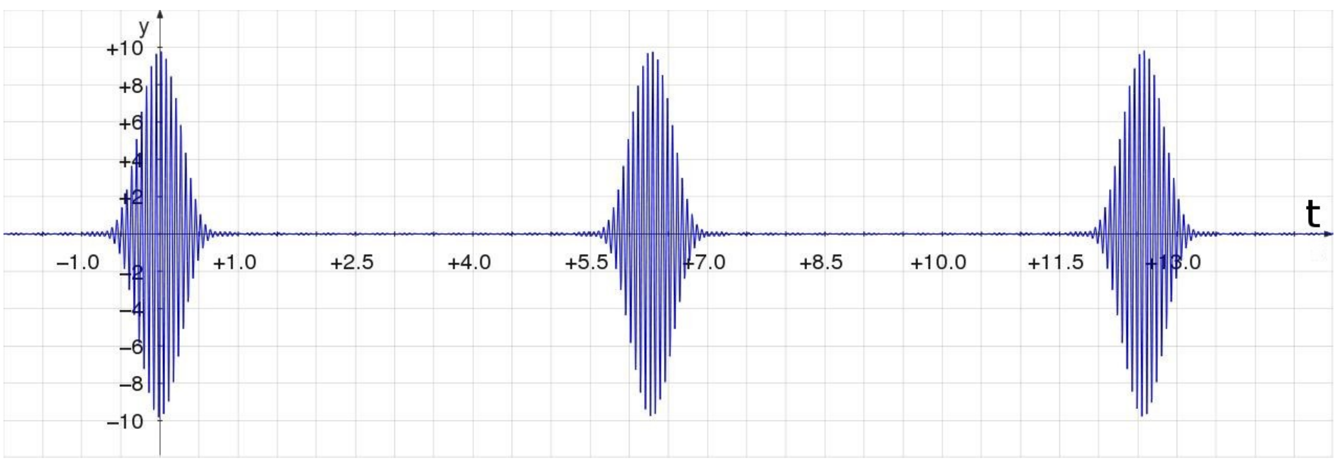
We are now in a position to get an idea how waves can produce particle-like behavior. Consider the 11-term series

$$y(t)=g(0)\sin(100t)+g(1)(\sin(99t)+\sin(101t))+g(2)(\sin(98t)+\sin(102t))$$
$$+g(3)(\sin(97t)+\sin(103t))+g(4)(\sin(96t)+\sin(104t))+g(5)(\sin(95t)+\sin(105t))$$

where $g(z)=e^{-\frac{1}{2}\left(\frac{z}{4}\right)^2}$ is a bell-shaped (Gaussian) curve with half-width of 4. Because of this function, the amplitude of the frequencies on either side of the central peak are reduced from 1 at the peak to 0.46 for the last term. One can think of *g(z)* as an envelope function controlling the strength of the component frequencies. This sum produces a series of pulses as shown below:



If this series is carried out for 19 terms where *g*(19)=0.000013, we get a more distinct pulse shape:

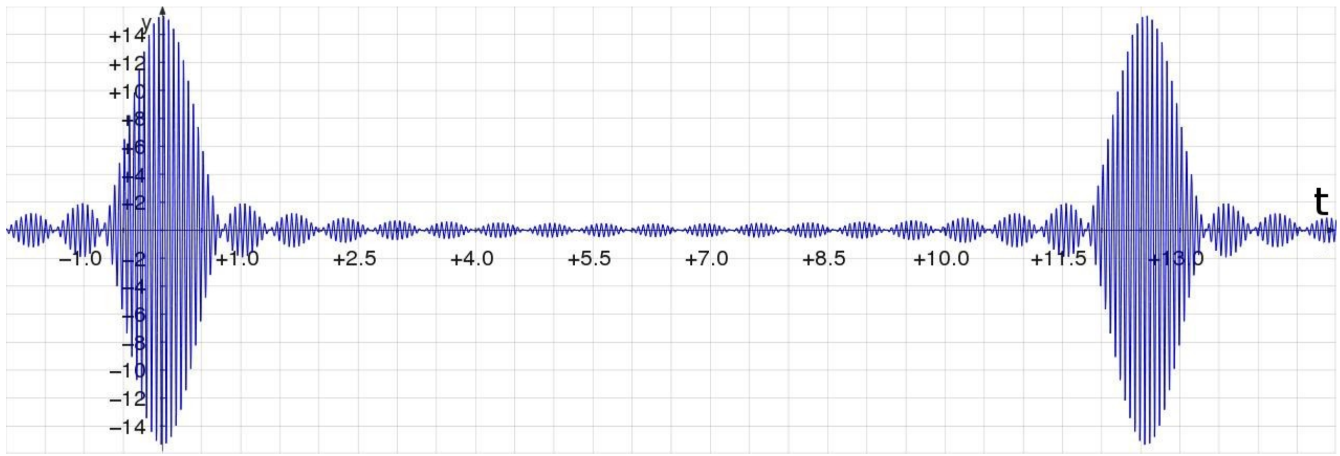


The more terms, the more perfect that cancellation between pulses and the smoother the pulse shapes.

Now, we can get pulses that are farther apart by using smaller differences between the frequencies. For example, if we use the 21-term series

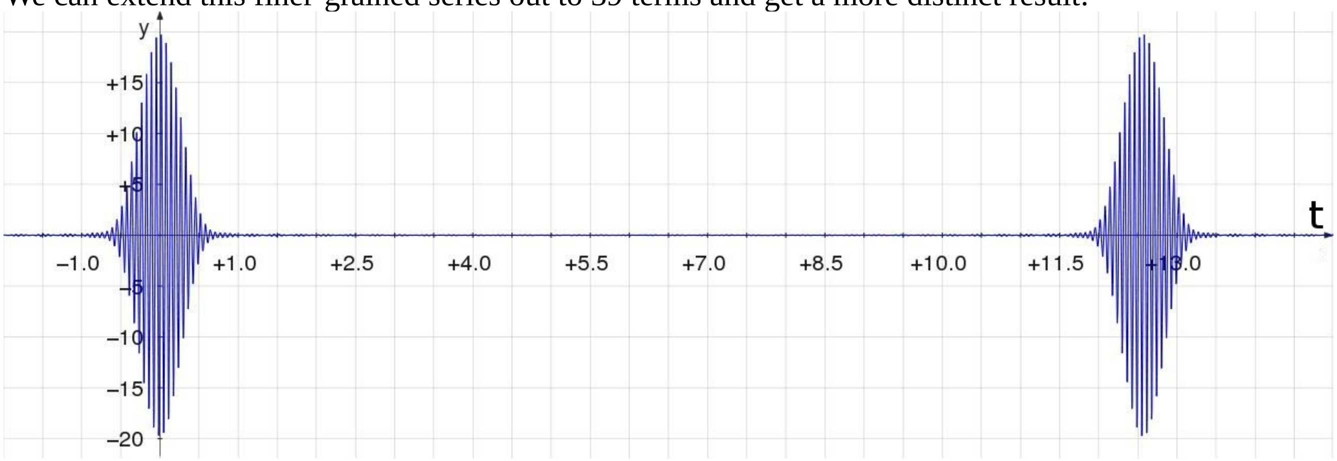
$$
\begin{aligned}
y(t) = g(0)\sin(100\mathrm{t}) &+ g(0.5)(\sin(99.5\mathrm{t})+\sin(100.5\mathrm{t})) + g(1)(\sin(99\mathrm{t})+\sin(101\mathrm{t})) \\
&+ g(1.5)(\sin(98.5\mathrm{t})+\sin(101.5\mathrm{t})) + g(2)(\sin(98\mathrm{t})+\sin(102\mathrm{t})) \\
&+ g(2.5)(\sin(97.5\mathrm{t})+\sin(102.5\mathrm{t})) + g(3)(\sin(97\mathrm{t})+\sin(103\mathrm{t})) \\
&+ g(3.5)(\sin(96.5\mathrm{t})+\sin(103.5\mathrm{t})) + g(4)(\sin(96\mathrm{t})+\sin(104\mathrm{t})) \\
&+ g(4.5)(\sin(95.5\mathrm{t})+\sin(104.5\mathrm{t})) + g(5)(\sin(95\mathrm{t})+\sin(105\mathrm{t}))
\end{aligned}
$$

We get the following pulse train:

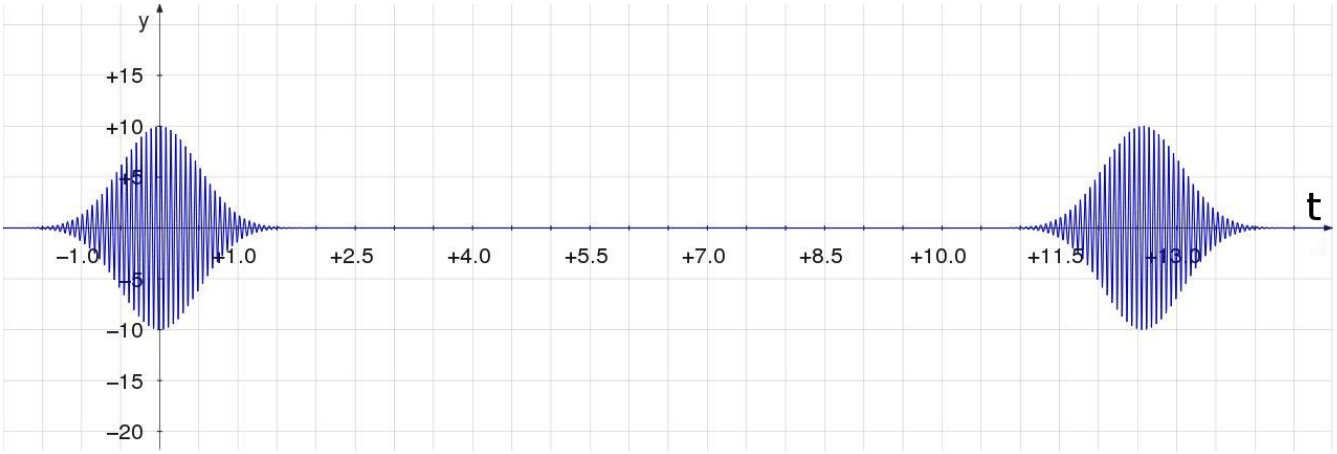


The pulses are now twice as far apart. **Thus, we see that the more closely spaced the component frequencies, the farther apart the pulses become vs. time. An infinitesimally close spacing of frequencies corresponds to an isolated pulse.**

We can extend this finer-grained series out to 39 terms and get a more distinct result:



If instead of using the envelope function *g(t)* with a half-width of 4, we sharpen it by using a half-width of 2. This last result becomes



The pulse width has doubled. **In general, the pulse width varies inversely with envelope function width. A narrow pulse width corresponds to broader spectrum of frequencies.**

When Wave Speed is Not Constant

So far, when assembling frequencies into waveforms, we have assumed that the wave speed $v=\omega/k$ is constant, independent of wavelength and amplitude, and position. This is the case for light waves in a vacuum. Under those assumptions, traveling wave packets move at the phase speed and retain their shape as they move along.

If, however, the wave speed varies with frequency as for light waves in glass, the wave shape will change, usually becoming less well-defined as some component frequencies move faster than others. A square pulse will lose its sharp edges and become smeared out in time. The wave envelope will also move slower than the wave speed so two separate velocities are involved, a phase velocity $v_p=\omega/k$ and a slower group velocity v_g. Only the group velocity can carry information and it will always be less than the speed of light in a vacuum. The phase velocity can carry no information and can be faster than the speed of light.

This affects optical fibers used in communication and is closely related to the Heisenberg Uncertainty principle in quantum mechanics.

Practical Problems Caused by Reflections and Standing Waves

Reflections and their associated standing waves are another problem that becomes important when the length of a wave transmission path is comparable or longer than the wave length of the waves being sent down the path. These can cause a loss of power transmission between end points since more current is flowing through the lines than is necessary. It can also cause communication signals to be echoed causing "ghosts" in television signals, loss of fidelity in music transmission, and bit errors in digital communications.

Early telegraph lines were limited in length by this problem until it was realized that a careful uniform geometry for the wires was needed to minimize the smearing and allow long-distance transmissions.

This is particularly important for high-frequency communication lines such as those used in cable television and DSL lines. The connection cables must have uniform characteristics along their length and the circuits at their end points must be carefully designed to minimize reflections.

Even though electrical power transmission lines operate at only 60 Hz and have a wavelength of several thousand kilometers, they are often hundreds of kilometers in length. When thousands of megawatts are sent down a transmission line, even a small amount of reflection can be a serious problem. Considerable effort is put into power transmission line and terminal equipment to minimize these reflections and resulting losses.

Modern computers operate at such high frequencies that their signals have wavelengths comparable in size to the motherboard. Motherboard designers must use traveling wave theory to correctly manage such problems.

Relation to Quantum Mechanics

In the limit of an infinite number of component frequencies that are spaced an infinitesimal distance from each other, the result will be an isolated pulse. This is how a particle is made from probability waves in quantum mechanics.

When the particle is a photon traveling in a vacuum, there is no dispersion. Its group and phase velocity are both equal to *c* which allow photons to travel across inter-galactic distances without distortion.

In quantum mechanics, particle momentum is connected to the wave number of probability waves and particle energy is connected to their frequency. The particle is a wave packet, but the waves making up the packet have significant dispersion. The particle group velocity is its velocity $v_g = v$, but its phase velocity is $v_p = c^2/v$. As a result, the shape of the particle will broaden as it travels through space. If the particle shape is sharply defined initially, later it will become broadened. The sharper (small δx) it is at first, the more rapidly (large δk and therefore large δp) it will become broadened. The Heisenberg Uncertainty Principle expresses this as $\delta x \cdot \delta p \geq \frac{h}{4\pi}$.

We have also seen that the spread of frequencies making up the pulse $\delta\omega$ were defined by our frequency envelope function, and that the sharper the envelope, the more spread out the pulse was in time δt. Even for particles (including photons), the spread in frequencies in the wave packet is connected to a spread of its energy. This produces another uncertainty relation $\delta E \cdot \delta t \geq \frac{h}{4\pi}$. If a photon has a narrow energy spread, its wave train will be stretched out over a long time, or if a photon is emitted over a long period of time, its energy spread (spectral line width) will be narrow.

Thus, there is nothing surprising about these uncertainty principles once one accepts that particles (light included) are waves.