Big and Small Numbers

2nd half of chessboard story:

http://en.wikipedia.org/wiki/Second_Half_of_the_Chessboard#Second_Half_of_the_Chessboard

When the creator of the game of chess showed his invention to the ruler of the country, the ruler was so pleased that he gave the inventor the right to name his prize for the invention. The man, who was very wise, asked the king this: that for the first square of the chess board, he would receive one grain of wheat, two for the second one, four on the third one and so forth, doubling the amount each time. The ruler, who was not strong in math, quickly accepted the inventor's offer, even getting offended by his perceived notion that the inventor was asking for such a low price, and ordered the treasurer to count and hand over the wheat to the inventor.

However, when the treasurer took more than a week to calculate the amount of wheat, the ruler asked him for a reason for his tardiness. The treasurer then gave him the result of the calculation, and explained that it would be impossible to give the inventor the reward.

The ruler then, to get back at the inventor who tried to outsmart him, told the inventor that in order for him to receive his reward, he was to count every single grain that was given to him, in order to make sure that the ruler was not stealing from him.

The amount of wheat is approximately 80 times what would be produced in one harvest, at modern yields, if all of Earth's arable land could be devoted to wheat.

In terms of volume: if we assume that a grain occupies a volume of 2 cubic millimeters, on average, then the total volume of all the grain on the chess board would be about 36.89 cubic kilometers.

The 0th through 40th powers of 2

2^0	=	1									
2^1	=	2	2^{11}	=	2,048	2^{21}	=	2,097,152	2^{31}	=	2,147,483,648
2^2	=	4	2^{12}	=	4,096	2^{22}	=	4,194,304	2^{32}	=	4,294,967,296
2^3	=	8	2^{13}	=	8,192	2^{23}	=	8,388,608	2^{33}	=	8,589,934,592
2^4	=	16	2^{14}	=	16,384	2^{24}	=	16,777,216	2^{34}	=	17,179,869,184
2^5	=	32	2^{15}	=	32,768	2^{25}	=	33,554,432	2^{35}	=	34,359,738,368
2^6	=	64	2^{16}	=	65,536	2^{26}	=	67,108,864	2^{36}	=	68,719,476,736
2^7	=	128	2^{17}	=	131,072	2^{27}	=	134,217,728	2^{37}	=	137,438,953,472
2^8	=	256	2^{18}	=	262,144	2^{28}	=	268,435,456	2^{38}	=	274,877,906,944
2^9	=	512	2^{19}	=	524,288	2^{29}	=	536,870,912	2^{39}	=	549,755,813,888
2^{10}	=	1,024	2^{20}	=	1,048,576	2^{30}	=	1,073,741,824	2^{40}	=	1,099,511,627,776

2^{63} = 9,223,372,036,854,775,808 grains on last square

2^{64} - 1 = 18,446,744,073,709,551,615 grains total for all squares

Different Ways of Counting:

We will use the following four ways of counting in the following notes:

1. Simply count: 1, 2, 3, 4, ...

This works fine when the number of items is not too large and each item can be individually identified. This method gives the exact count.

We will use this method with a group of ABC blocks.

2. By Weight:

Determine the average weight of a few of the items, and divide the average weight into the total weight of all items.

This may not get the exact count depending on the uniformity of the items and the accuracy of the weighing, but it is a useful way to estimate the count for a large numbers of items.

We will use this method for a jar of marbles and a jar of salt.

3. By Volume:

Determine the average volume of a few of the items, and divide the average volume into the total volume of all items.

Usually there will be empty space between the items which must be taken into consideration. This also may not get the exact count depending on the uniformity of the items, the accuracy of the volume measurement, accuracy of the empty space correction. Still, under some conditions, it is a useful way to estimate the count for a large numbers of items.

We will also use this method for the marbles and salt.

4. By Tagging:

Sample, tag, mix back, sample again and use a proportion calculation.

This is often used for wildlife population surveys, such as estimating the number of fish in a lake. It is illustrated by the following example:

Step 1: You catch some of the fish, count them, and put tags on their fins to identify that they are ones in this first catch. This will be the "*number of tagged fish in lake*".

Step 2: Put the tagged fish back into the lake and wait a few days for them to mix up with the other fish in the lake.

Step 3: Make a second catch of fish and count all of them, and also count the number of them that have tags. This gives you the remaining two items in your proportion problem (see step 4).

Step 4: Solve the proportion problem:

$$\frac{number\ of\ all\ fish\, in\, lake}{number\ of\ tagged\ fish\, lake} = \frac{number\ of\ all\ fish\, in\, step\, 3}{number\ of\ tagged\ fish\, in\, step\, 3}$$

We will use this with the marbles, simulating the tagging by exchanging colored marbles for some of the clear marbles.

Counting ABC blocks by weight and size (true count = 24):

Note:	'gram' is abbreviated by the single letter 'g' 'kilogram' is abbreviated by the letters 'kg' 1 kg = 1000 g

Note:	'milliliter' is abbreviated by the letters 'mL' 'liter' is abbreviated by 'L' 1 L= 1000 mL 'cubic centimeter' is abbreviated by the letters 'cc' 1 cc = 1 mL

Note:	'meter' is abbreviated the letter 'm' 'centimeter' is abbreviated by the letters 'cm' 'millimeter' is abbreviated by the letters 'mm' 1 m = 100 cm 1 cm = 10 mm 1 m = 1000 mm

Estimating count by weight:

Weight of stack of all blocks: 935 g
Weight of Red/Blue Fish/Frog Block: 42.56 g
Estimate of count = 935 g/42.56 g = 21.97 blocks which rounds to 22 blocks.

Weights of 4 randomly selected blocks:44.03 g, 41.99 g, 35.42 g, 33.55 g
Average Weight of those 4 blocks: 154.99 g/4 = 38.7475 g which rounds to 38.7 g
New estimate of count = 935 g/38.7 g = 24.16 blocks which rounds to 24 blocks.

Weights of all blocks:

33.56, 44.06, 39.48, 35.97, 38.00, 35.38, 35.72, 36.06, 40.67, 35.46, 52.54, 37.83, 44.78, 42.57, 38.78, 43.79, 38.76, 39.33, 36.23, 39.13, 30.37, 42.00, 43.46, 31.95

Average Weight of all blocks = 38.995 g
Still better estimate of count = 935 g/38.995 g = 23.98 blocks which rounds to 24 blocks.

Estimating count by volume:

Volume of stack of all blocks: $13.224\,\text{cm} \times 8.844\,\text{cm} \times 17.50\,\text{cm} = 2046.67848\,\text{cm}^3$
which rounds to 2046.7 cubic cm = 2046.7 mL
Volume of Fish/Frog Block: $4.335\,\text{cm} \times 4.372\,\text{cm} \times 4.408\,\text{cm} = 83.543\,\text{cm}^3 = 83.543\,\text{mL}$
Estimate of count = 2046.7 mL/83.543 mL = 24.499 blocks which rounds to 24 blocks.

Counting marbles by weight and volume (true count = 355):

Estimating count by weight:

Weight of all marbles: 1366 g

Weight of 25 marbles selected randomly: 96.37 g

Average weight of marble: $W_{marble} = 96.37\ \mathrm{g}/25 = 3.855\ \mathrm{g}$

Estimate of count = 1366 g/3.855 g = 354.35 marbles which rounds to 354

Estimating count by volume:

I bought 3 pounds of clear glass marbles at JoAnne's ($6.99) and measured a few getting the following results in inches (I had not yet bought metric vernier calibers.):

.569	.561	.561	.559	.560	.558	.559	.569	.561
.549	.563	.563	.565	.566	.555	.550	.575	.567
.574	.556	.566	.564	.555	.561	.559	.561	.569
.561	.576	.565	.557	.569	.571	.565	.563	
.562	.572	.557	.557	.583	.565	.561	.556	

These average to 0.563" with a standard deviation of 0.007".

Each was spherical to with a standard deviation of about 0.002".

Note: 2.54 cm = 1.000 inches
25.4 mm = 1.000 inches

We can now calculate the average volume of each marble:

$$V_{\text{marble}} = \frac{4}{3} \times \pi \times \left(\frac{0.563\ \text{inches} \times 2.54\ \text{cm}/\text{inch}}{2} \right)^3 = 1.533\ \text{cm}^3 = 1.533\ \text{mL}$$

But the marbles in the jar have some empty space between them. A better volume to use for each marble is probably a cube with sides equal to the marble diameter. This cube would have a volume of

$$V_{\text{marble in cube}} = (0.563\ \text{inches} \times 2.54\ \text{cm}/\text{inch})^3 = 2.92\ \text{cm}^3 = 2.92\ \text{mL}$$

Putting all of the marbles into a measuring beaker shows that they take up pretty close to 1000 mL of volume. We can therefore estimate the marble count to be

1000 mL / 2.92 mL = 342.46 marbles which rounds to 342 marbles

The exact count is 355.

Note: The density of glass is can be obtained from our values of marble weight and marble volume.

$$\rho_{\text{glass}} = \frac{W_{\text{marble}}}{V_{\text{marble}}} = \frac{3.855\ \text{g}}{1.533\ \text{mL}} = 2.515\ \text{g}/\text{mL} = 2.515\ \text{g}/\text{cm}^3$$

The value in wikipedia for the density of glass is 2.52 g/cc

Salt grains in jar:

Estimate by weight:

0.10 g of salt was counted and found to contain about 1000 grains.

So 1 g would contain about 10,000 grains which can be written as $10{,}000\,\frac{\text{grains of salt}}{\text{g of salt}}$

The salt in our jar weighs 618 g, so our jar must have about

$$10{,}000\,\frac{\text{grains of salt}}{\text{g of salt}}\times 618\text{ g of salt} = 6{,}180{,}000\text{ grains of salt}$$

Estimate by volume:

The salt in our jar has a volume of 460 mL, and a typical grain is a cube of about 0.03 cm on a side. That grain then has a volume of

$$V_{\text{grain of salt}} = (0.03\text{ cm})^3 = 0.000027\text{ cm}^3 = 0.000027\text{ mL}$$

But the salt grains are not tightly packed. Each grain might actually take up about 0.00004 mL.

This means there are $\frac{1}{0.00004\text{ mL}} = 25{,}000$ grains per mL

Using this value, we can estimate the number of grains in the jar to be

$$460\text{ mL}\times 25000\text{ grains per mL} = \mathbf{11{,}500{,}00\ grains\ of\ salt\ in\ jar}$$

Scientific Notation for Large Numbers

Scientists have a very useful shorthand way to do arithmetic with large numbers.
First notice the connection between powers of 10 and numbers that have a 1 followed by many zeros:

$100 = 10\times 10 = 10^2$	2 zeros in the number, 10 to the 2nd power
$1000 = 10\times 10\times 10 = 10^3$	3 zeros in the number, 10 to the 3rd power
$10{,}000 = 10\times 10\times 10\times 10 = 10^4$	4 zeros in the number, 10 to the 4th power
$100{,}000 = 10\times 10\times 10\times 10\times 10 = 10^5$	5 zeros in the number, 10 to the 5th power
$1{,}000{,}000 = 10\times 10\times 10\times 10\times 10\times 10 = 10^6$	6 zeros in the number, 10 to the 6th power
...	
$1{,}000{,}000{,}000{,}000{,}000{,}000 = 10^{18}$	18 zeros in the number, 10 to the 18th power

And then notice what happens when we multiply two such numbers:

$$10{,}000\times 1{,}000{,}000{,}000{,}000{,}000{,}000 = 10{,}000{,}000{,}000{,}000{,}000{,}000{,}000 = 10^4\times 10^{18} = 10^{4+18} = 10^{22}$$

Now notice the following:

$$6{,}000{,}000{,}000{,}000{,}000{,}000 = 6\times 1{,}000{,}000{,}000{,}000{,}000{,}000 = 6\times 10^{18}$$

And finally notice:

$$30{,}000{,}000\times 6{,}000{,}000{,}000{,}000{,}000{,}000 = 3\times 10^7\times 6\times 10^{18} = 3\times 6\times 10^{7+18} = 18\times 10^{25}$$

Salt grains in classroom:

What if the entire classroom were filled with salt?
If the classroom were 15 m wide by 10 m long by 3 meters tall, its volume would be $15 \times 10 \times 3 = 450\ \text{m}^3$.
Each cubic meter is $100\ \text{cm} \times 100\ \text{cm} \times 100\ \text{cm} = 1{,}000{,}000\ \text{cm}^3 = 1.0 \times 10^6\ \text{cm}^3 = 1.0 \times 10^6\ \text{mL}$.
So the 450-cubic meter classroom would contain $450 \times 1.0 \times 10^6\ \text{mL} = 4.5 \times 10^8\ \text{mL}$.
And since there are 25000 grains of salt in each mL, we would have

$$25000\ \text{grains of salt/mL} \times 4.5 \times 10^8\ \text{mL} = 1.125 \times 10^{13}\ \text{grains of salt}$$.

To keep things simple we will round this to simply $1.0 \times 10^{13}\ \text{grains of salt}$.

10,000,000,000,000 grains of salt in a classroom

Salt grains in a box the size of Raymond:

The town of Raymond is about 1000 m long by 1000 m wide. If we made a box that size that was 40 m high, it would contain $4.0 \times 10^7\ \text{m}^3 = 4.0 \times 10^{13}\ \text{cm}^3 = 4.0 \times 10^{13}\ \text{mL}$.
So the number of grains of salt in our Raymond-sized box would be

$$4.0 \times 10^{13}\ \text{mL} \times 25000\ \text{grains of salt/mL} = 1.0 \times 10^{18}$$

1,000,000,000,000,000,000 grains of salt in Raymond-sized box

Molecules in salt grain:

Long ago scientists figured out that one gram of salt (NaCl) has about 1.0×10^{22} molecules.
Earlier we found that 1 g of salt contained about $10000\ \text{grains} = 1.0 \times 10^4\ \text{grains}$.
So a single grain of salt has about $\dfrac{1.0 \times 10^{22}\ \text{molecules/g}}{1.0 \times 10^4\ \text{grains/g}} = 1.0 \times 10^{18}\ \text{molecules/grain}$

1,000,000,000,000,000,000 molecules of salt in one grain of salt

Tagging Marbles:

We took some marbles from our jar, counted them, and got a count of 20.

We simulated tagging these marbles by replacing those 20 with 20 colored marbles.

We then put the "tagged" marbles back in the jar.

We shook the jar to thoroughly mix all marbles.

Finally, we grabbed a handful of marbles without looking. When we checked our catch, we found that we had 3 colored marbles out of 28 total in our catch.

We now have the data necessary to do a proportion calculation to get an estimate for the number of marbles in the jar:

$$\frac{x}{20}=\frac{28}{3}$$

So our estimate for the number of marbles in the jar is $x=\frac{20\times 28}{3}=186.6$ which rounds to 187.

Remember, the true answer was 255. Our answer would be closer if we "tagged" more than only 20 marbles or later caught more than 28.

Other Large Numbers:

(http://en.wikipedia.org/wiki/Large_numbers)

The universe seems to be about 4.3×10^{17} seconds old (13.7×10^{9} years).
The observable universe is 4.65×10^{26} meters across (93×10^{9} light years)
and contains about 5×10^{22} stars organized in about 1.25×10^{11} galaxies
Light travels at 3×10^{8} meters per second.
The number of cells in the human body is more than 10^{14} .
The number of neuronal connections in the human brain is about 10^{14} .
The number of atoms in 12 grams of carbon (Avogadro's number) is about 6.022×10^{23} .
The mass of the earth is 5.9736×10^{27} grams.